Fractional Hamilton formalism within Caputo's derivative

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Abstract

In this paper we develop a fractional Hamiltonian formulation for dynamic systems defined in terms of fractional Caputo derivatives. Expressions for fractional canonical momenta and fractional canonical Hamiltonian are given, and a set of fractional Hamiltonian equations are obtained. Using an example, it is shown that the canonical fractional Hamiltonian and the fractional Euler-Lagrange formulations lead to the same set of equations.

1 Introduction

Fractional calculus is one of the generalizations of the classical calculus and it has been used successfully in various fields of science and engineering [1-10]. A huge amount of mathematical knowledge on fractional integrals and derivatives has been constructed [1-8]. The fractional derivatives represent the infinitesimal generators of a class of translation invariant convolution semigroups which appear universally as attractors.

During the last decade several papers, which deal with fractional variational calculus and its applications, have been published [11-30]. These applications include classical and quantum mechanics, field theory, and optimal control. Due the properties of the fractional derivatives the corresponding theories become non-local and non-commutative. In the papers cited above, the problems have been formulated mostly in terms of two types of fractional derivatives, namely Riemann-Liouville (RL) and Caputo. Among mathematicians, RL fractional derivatives have been popular largely because they are amenable to many mathematical manipulations. However, the RL derivative of a constant is not zero, and in many applications it requires fractional initial conditions which are generally not specified. Many believe that

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fractional initial conditions are not physical. In contrast, Caputo derivative of a constant is zero, and a fractional differential equation defined in terms of Caputo derivatives require standard boundary conditions. For these reasons, Caputo fractional derivatives have been popular among engineers and scientists.

Recently, Agrawal [20] demonstrated that fractional terminal conditions may be necessary even when a problem is formulated in terms of Caputo derivatives. In [20] it also argued that fractional initial conditions may have physical meaning. For example, in a one dimensional heat diffusion process 1/2 order time derivative of temperature may represent heat flux across a surface. This blurs the distinctions between advantages and disadvantages of RL and Caputo derivatives. Thus, which derivative would be more suitable for formulating engineering and scientific problems remain an open issue.

Recently, the fractional Hamiltonian formulations were presented for fractional discrete and continuous systems whose dynamics were defined in terms of RL derivatives [20-27]. In this paper we construct a fractional Hamiltonian formulation for discrete systems whose dynamics have been described using Caputo derivatives. It is demonstrated in [20] that the Euler-Lagrange equation of a fractional variational problem defined in terms of Caputo derivatives only includes both the RL and the Caputo derivatives. The same is true of the fractional Hamiltonian formulation discussed here. An example is solved in detail to demonstrate an application of the Formulation.

2 Mathematical Tools

In this section we briefly present some fundamental definitions used in the previous section. The left and the right Riemann-Liouville and Caputo fractional derivatives are defined as follows:

The Left Riemann-Liouville Fractional Derivative

$$ {}_aD_x^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_a^x (x-\tau)^{n-\alpha-1} f(\tau) d\tau, \tag{1}$$

The Right Riemann-Liouville Fractional Derivative

$$ {}_xD_b^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \left(-\frac{d}{dx}\right)^n \int_x^b (\tau-x)^{n-\alpha-1} f(\tau) d\tau, \tag{2}$$

The corresponding Caputo's fractional derivatives are defined as follows

The Left Caputo Fractional Derivative

$$
{}_{a}^{C}D_{x}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{a}^{x}(x-\tau)^{n-\alpha-1}\left(\frac{d}{d\tau}\right)^{n}f(\tau)d\tau, \tag{3}
$$

and

The Right Caputo Fractional Derivative

$$
{}_{x}^{C}D_{b}^{\alpha}f(x)=\frac{1}{\Gamma(n-\alpha)}\int_{x}^{b}(\tau-x)^{n-\alpha-1}\left(-\frac{d}{d\tau}\right)^{n}f(\tau)d\tau, \tag{4}
$$

where α is the order of the derivative such that $n-1<\alpha<n$. These derivatives will be denoted as the LRLFD, the RRLFD, the LCFD, and the RCFD, respectively.

Our fractional Hamiltonian formulation presented here is based on the following theorem [20]. This theorem is stated here (without proof) for completeness.

Theorem *Let $J[q]$ be a functional of the form*

$$
J[q]=\int_{a}^{b}L(t,q,\,{}_{a}^{C}D_{t}^{\alpha}q,\,{}_{t}^{C}D_{b}^{\beta}q)dt \tag{5}
$$

where $0<\alpha,\beta<1$ *and defined on the set of functions $y(x)$ which have continuous LCFD of order α and RCFD of order β in $[a,b]$. Then a necessary condition for $J[q]$ to have an extremum for a given function $q(t)$ is that $q(t)$ satisfy the generalized Euler-Lagrange equation given by*

$$
\frac{\partial L}{\partial q}+{}_{t}D_{b}^{\alpha}\frac{\partial L}{\partial\,{}_{a}^{C}D_{t}^{\alpha}q}+{}_{a}D_{t}^{\beta}\frac{\partial L}{\partial\,{}_{t}^{C}D_{b}^{\beta}q}=0,\qquad t\in[a,\ b] \tag{6}
$$

and the transversality conditions given by

$$
\left[{}_{t}D_{b}^{\alpha-1}\left(\frac{\partial L}{\partial\,{}_{a}^{C}D_{t}^{\alpha}q}\right)-{}_{a}D_{t}^{\beta-1}\left(\frac{\partial L}{\partial\,{}_{t}^{C}D_{b}^{\beta}q}\right)\right]\eta(t)|_{a}^{b}=0. \tag{7}
$$

The proof of the theorem can be found in [20].

3 Fractional Hamilton formulation

In this section we construct the Hamiltonian formulation within Caputo's fractional derivatives.

Let us consider the fractional Lagrangian as given below

$$L(q, {}_a^C D_t^\alpha q, {}_t^C D_b^\beta q, t), \quad 0 < \alpha, \beta < 1. \tag{8}$$

By using (8) we define the canonical momenta p_α and p_β as follows

$$p_\alpha = \frac{\partial L}{\partial {}_a^C D_t^\alpha q}, \quad p_\beta = \frac{\partial L}{\partial {}_t^C D_b^\beta q}. \tag{9}$$

Making use of (8) and (9) we define the fractional canonical Hamiltonian as

$$H = p_\alpha \, {}_a^C D_t^\alpha q + p_\beta \, {}_t^C D_b^\beta q - L. \tag{10}$$

Taking total differential of (10) and by using (9), we obtain

$$dH = dp_\alpha \, {}_a^C D_t^\alpha q + dp_\beta \, {}_t^C D_b^\beta q - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt. \tag{11}$$

Taking into account the fractional Euler-Lagrange equations (6) we obtain

$$dH = dp_\alpha \, {}_a^C D_t^\alpha q + dp_\beta \, {}_t^C D_b^\beta q + ({}_t D_b^\alpha p_\alpha + {}_a D_t^\beta p_\beta) dq - \frac{\partial L}{\partial t} dt. \tag{12}$$

Finally, after some simple manipulations the fractional Hamilton equations are obtained as follows

$$\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}, \;\; \frac{\partial H}{\partial p_\alpha} = {}_a^C D_t^\alpha q, \;\; \frac{\partial H}{\partial p_\beta} = {}_t^C D_b^\beta q, \;\; \frac{\partial H}{\partial q} = {}_t D_b^\alpha p_\alpha + {}_a D_t^\beta p_\beta. \tag{13}$$

In the following, an example is considered to demonstrate an application these equations.

3.1 Example

Consider the following problem [20]: Minimize

$$J[q] = \frac{1}{2} \int_0^1 ({}_0^C D_t^\alpha q - \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{(\beta-\alpha)})^2 dt, \qquad 0 < \alpha < 1, \tag{14}$$

such that $q(0) = 0,$ and $q(1) = 1$. For this example, the Lagrangian is given by

$$L = ({}_0^C D_t^\alpha q - \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{(\beta-\alpha)})^2. \tag{15}$$

The corresponding Euler-Lagrange equation has the form

$${}_t D_1^\alpha ({}_0^C D_t^\alpha q \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{(\beta-\alpha)}) = 0, \tag{16}$$

and admits the solution as

$$q(t) = t^\beta. \tag{17}$$

The fractional canonical momenta is given by

$$p_\alpha = {}_0^C D_t^\alpha q - \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{(\beta-\alpha)} \tag{18}$$

and the fractional canonical Hamiltonian is given by

$$H = p_\alpha {}_0^C D_t^\alpha q - L \tag{19}$$

or

$$H = \frac{p_\alpha^2}{2} + p_\alpha \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{(\beta-\alpha)}. \tag{20}$$

The canonical equations of motion are given by

$$\frac{\partial H}{\partial p_\alpha} = {}_a^C D_t^\alpha q, \quad \frac{\partial H}{\partial q} = {}_t D_b^\alpha p_\alpha \tag{21}$$

or

$$p_\alpha + \frac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)} t^{(\beta-\alpha)} = {}_0^C D_t^\alpha q, {}_t D_1^\alpha p_\alpha = 0. \tag{22}$$

By inspection we observed that (22) is equivalent with (16).

4 Conclusion

Fractional calculus was intensively applied, especially during the last decade, for describing the dynamics of the complex systems. The canonical fractional formulation is still an open problem of this new and emerging field.

The characteristic of the constructed fractional Hamilton equations within Caputo's derivatives is that both RL and Caputo's fractional derivatives are involved. Generally, to find a solution for the system (13) we have to replace RL fractional derivative in terms of Caputo's fractional derivative and then to solve the corresponding system.

For a given mechanical example we observed that both fractional Euler-Lagrange equations and fractional Hamiltonian equations give the same result. The classical results are obtained as a particular case of the fractional formulation.

The fractional Lagrangians and their corresponding fractional Hamiltonians are non-local. This property comes from the definitions of the fractional derivatives, therefore finding the physical interpretation of these derivative is an open issue.

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